Det Kgl. Danske Videnskabernes Selskab.
Mathematatisk-fysiske Meddelelser. **XVII,** 4.

FOURIER EXPANSIONS FOR PERIODIC ORBITS AROUND THE TRIANGULAR LIBRATION POINTS

BY

PEDER PEDERSEN

KØBENHAVN
EJNAR MUNKSGAARD
1939

Printed in Denmark
Bianco Lunos Bogtrykkeri A/S

In two papers published in the Monthly Notices of the Royal Astronomical Society I have treated the Fourier expansions for the infinitesimal periodic orbits around the triangular libration points in the restricted problem of three bodies. In the first paper[1] I considered the special case that the mass product deviates only slightly from the critical value $\frac{1}{27}$, while the general case was treated in the second paper[2]. In the series expansions of the differential equations of motion terms of the third order were retained, so that the coefficients in the Fourier series of the co-ordinates could be determined with third order accuracy.

It followed from the investigation that the Fourier series for the short-period orbits could always be determined, whereas for certain critical values of the mass product the Fourier series for the long-period orbits could not be found.

When the value of the mass product is close to $\frac{1}{27}$, however, a difficulty of a special nature arises, namely, that the Fourier series exist, but are of a form that differs from the normal. In my first paper the investigation of these series was not carried completely through. As it seemed to be a matter of considerable interest to clear up the problems of the periodic orbits near the triangular libration points for values of the mass product close to $\frac{1}{27}$,

[1] M. N., 94, 167—185, 1933. Publ. fra Københavns Obs. Nr. 91.
[2] M. N., 95, 482—495, 1935. Publ. fra Københavns Obs. Nr. 101.

I have carried out the complete calculation of the Fourier series up to terms of the third order, using my first paper as a basis.

In what follows all references are to my first paper in the Monthly Notices.

The co-ordinates ξ and η of the infinitesimal mass are to be represented by the Fourier series

$$\left.\begin{aligned}\xi &= a_0 + a_1 \cos\omega t + a_{-1}\sin\omega t\\ &\quad + a_2\cos 2\,\omega t + a_{-2}\sin 2\,\omega t\\ &\quad + a_3\cos 3\,\omega t + a_{-3}\sin 3\,\omega t,\\ \eta &= b_0 + b_1\cos\omega t + b_{-1}\sin\omega t\\ &\quad + b_2\cos 2\,\omega t + b_{-2}\sin 2\,\omega t\\ &\quad + b_3\cos 3\,\omega t + b_{-3}\sin 3\,\omega t,\end{aligned}\right\} \qquad \begin{matrix}(1)\\ [\text{M.N.}\ (14)]\end{matrix}$$

where the constants of integration are so chosen that

$$a_1 = \varepsilon, \qquad b_1 = 0. \qquad \begin{matrix}(2)\\ [\text{M.N.}\ (29, 30)]\end{matrix}$$

The other coefficients in the series are then functions of ε.

When terms up to the 3. order in the series expansions are retained, the equation that determines ω takes the form

$$\left(\omega^2 - \frac{1}{2}\right)^2 + \frac{27}{4}\,\delta - \frac{295}{864}\,\varepsilon^2 = 0, \qquad \begin{matrix}(3)\\ [\text{M.N.}\ (63)]\end{matrix}$$

where δ denotes the deviation of the mass product from $\frac{1}{27}$, thus

$$\mu\,(1-\mu) = \frac{1}{27} + \delta. \qquad \begin{matrix}(4)\\ [\text{M.N.}\ (45)]\end{matrix}$$

Assuming ε to be a small quantity of the 1. order, we find that $\omega^2 - \frac{1}{2}$ is also of the 1. order, while δ is a small quantity of the 2. order.

In the general case where δ is finite, ω can be determined by an equation corresponding to (3) with 2. order accuracy. The coefficients in the Fourier expansions can then be found with 3. order accuracy. In the special case here considered, however, δ is assumed to be a small quantity of the 2. order. It is seen that in this case ω is determined by equation (3) with 1. order accuracy only. In consequence the Fourier coefficients, with the exception of the coefficients of the 3. order a_3, a_{-3}, b_3 and b_{-3}, can only be found up to terms of the 2. order.

In order to determine the coefficients up to terms of the 3. order even in the special case, it is necessary to retain, in the original series expansions, terms up to the 4. order. A detailed investigation now shows that the coefficient scheme previously used [M.N., 172] suffices even in the case that terms up to the 4. order have to be retained. It is necessary, however, to recalculate the quantities in the coefficient scheme.

Let ω_0 denote the value of ω that satisfies equation (3). For the sake of brevity we introduce

$$\omega_0^2 - \frac{1}{2} = \zeta_0, \tag{5}$$

so that equation (3) now becomes

$$\zeta_0^2 + \frac{27}{4}\delta - \frac{295}{864}\varepsilon^2 = 0. \tag{6}$$

In this equation ε and ζ_0 are of the 1. order, while δ is of the 2. order.

We shall now express the coefficients in the Fourier series as functions of ε and ζ_0. From equation (5) we get

$$\omega_0 = \frac{1}{2}\sqrt{2}\,(1+\zeta_0). \tag{7}$$

We now see that ω, as was already mentioned, is for the present determined up to terms of the 1. order only. Introducing ω_0 instead of ω in the equations [M.N.(35)], leaving out terms of the 3. order for the present, we get

$$\left.\begin{aligned} a_{-1} &= \frac{3}{8}\sqrt{6}\,k\,\varepsilon\,(1-\zeta_0), \\ b_{-1} &= -\frac{5}{8}\sqrt{2}\,\varepsilon\left(1-\frac{1}{5}\zeta_0\right). \end{aligned}\right\}\left(k=\sqrt{\frac{23}{27}}\right) \tag{8}$$

The values of a_{-1} and b_{-1} are now determined up to terms of the 2. order. With these values it is now possible to recompute the quantities in the coefficient scheme to the accuracy required.

In the determination of a_0 and b_0 we use the first column of the coefficient scheme. Inserting the values of a_1, b_1, a_{-1} and b_{-1} we get the values given in the scheme.

ξ^2	$\frac{55}{64}\varepsilon^2 - \frac{23}{32}\zeta_0\varepsilon^2$
$\xi\eta$	$-\frac{15}{64}\sqrt{3}\,k\,\varepsilon^2 + \frac{9}{32}\sqrt{3}\,k\,\zeta_0\varepsilon^2$
η^2	$\frac{25}{64}\varepsilon^2 - \frac{5}{32}\zeta_0\varepsilon^2$

(9)

With $\zeta_0 = 0$ the scheme reduces to the scheme [M.N. (53)]. For the determination of a_0 and b_0 we have the following two equations:—

$$\left.\begin{aligned} \frac{3}{4} a_0 + \frac{3}{4}\sqrt{3}\,k b_0 &= -\frac{75}{128} k\varepsilon^2 + \frac{15}{16} k\zeta_0\varepsilon^2, \\ \frac{3}{4}\sqrt{3}\,k a_0 + \frac{9}{4} b_0 &= -\frac{85}{192}\sqrt{3}\,\varepsilon^2 + \frac{49}{64}\sqrt{3}\,\zeta_0\varepsilon^2. \end{aligned}\right\} \quad (10)$$

These equations have the roots

$$\left.\begin{aligned} a_0 &= -\frac{165}{128} k\varepsilon^2 + \frac{99}{64} k\zeta_0\varepsilon^2, \\ b_0 &= \frac{65}{384}\sqrt{3}\,\varepsilon^2 - \frac{19}{192}\sqrt{3}\,\zeta_0\varepsilon^2. \end{aligned}\right\} \quad (11)$$

Now the quantities in the fourth and fifth column of the coefficient scheme are determined. We thus get:—

	$\cos 2\omega t$	$\sin 2\omega t$
ξ^2	$\frac{9}{64}\varepsilon^2 + \frac{23}{32}\zeta_0\varepsilon^2$	$\frac{3}{8}\sqrt{6}\,k\varepsilon^2 - \frac{3}{8}\sqrt{6}\,k\zeta_0\varepsilon^2$
$\xi\eta$	$\frac{15}{64}\sqrt{3}\,k\varepsilon^2 - \frac{9}{32}\sqrt{3}\,k\zeta_0\varepsilon^2$	$-\frac{5}{16}\sqrt{2}\,\varepsilon^2 + \frac{1}{16}\sqrt{2}\,\zeta_0\varepsilon^2$
η^2	$-\frac{25}{64}\varepsilon^2 + \frac{5}{32}\zeta_0\varepsilon^2$	0

(12)

With $\zeta_0 = 0$ the scheme reduces to the scheme [M.N. (56)]. For the determination of a_2, a_{-2}, b_2 and b_{-2} we have the equations

$$
\left.\begin{aligned}
&\left(4\,\omega^2+\frac{3}{4}\right)a_2 + \frac{3}{4}\sqrt{3}\,k\,b_2 + 4\,\omega\,b_{-2} = \\
&\qquad = -\frac{93}{128}\,k\,\varepsilon^2 - \frac{15}{16}\,k\,\zeta_0\,\varepsilon^2,\\
&\frac{3}{4}\sqrt{3}\,k\,a_2 + \left(4\,\omega^2+\frac{9}{4}\right)b_2 - 4\,\omega\,a_{-2} = \\
&\qquad = \frac{121}{192}\sqrt{3}\,\varepsilon^2 - \frac{49}{64}\sqrt{3}\,\zeta_0\,\varepsilon^2,\\
&-4\,\omega\,b_2 + \left(4\,\omega^2+\frac{3}{4}\right)a_{-2} + \frac{3}{4}\sqrt{3}\,k\,b_{-2} = \\
&\qquad = -\frac{103}{192}\sqrt{6}\,\varepsilon^2 + \frac{85}{192}\sqrt{6}\,\zeta_0\,\varepsilon^2,\\
&4\,\omega\,a_2 + \frac{3}{4}\sqrt{3}\,k\,a_{-2} + \left(4\,\omega^2+\frac{9}{4}\right)b_{-2} = \\
&\qquad = -\frac{69}{64}\sqrt{2}\,k\,\varepsilon^2 + \frac{3}{64}\sqrt{2}\,k\,\zeta_0\,\varepsilon^2.
\end{aligned}\right\} \quad (13
$$

The roots are

$$
\left.\begin{aligned}
a_2 &= -\frac{11}{128}\,k\,\varepsilon^2 - \frac{1}{192}\,k\,\zeta_0\,\varepsilon^2,\\
b_2 &= \frac{79}{3456}\sqrt{3}\,\varepsilon^2 + \frac{841}{5184}\sqrt{3}\,\zeta_0\,\varepsilon^2,\\
a_{-2} &= -\frac{127}{864}\sqrt{6}\,\varepsilon^2 + \frac{1381}{2592}\sqrt{6}\,\zeta_0\,\varepsilon^2,\\
b_{-2} &= -\frac{13}{96}\sqrt{2}\,k\,\varepsilon^2 - \frac{29}{288}\sqrt{2}\,k\,\zeta_0\,\varepsilon^2.
\end{aligned}\right\} \quad (14)
$$

With the values of a_0, b_0, a_2, a_{-2}, b_2 and b_{-2} thus found it is possible to calculate the quantities in the second and third column of the coefficient scheme up to terms of the 4. order. The values given below result:—

	$\cos \omega t$	$\sin \omega t$
ξ^2	$-\frac{575}{192} k \varepsilon^3 + \frac{665}{144} k \zeta_0 \varepsilon^3$	$-\frac{26075}{27648} \sqrt{6}\, \varepsilon^3 + \frac{192335}{82944} \sqrt{6}\, \zeta_0 \varepsilon^3$
$\xi \eta$	$\frac{1585}{6912} \sqrt{3}\, \varepsilon^3 - \frac{3713}{10368} \sqrt{3}\, \zeta_0 \varepsilon^3$	$\frac{455}{512} \sqrt{2}\, k \varepsilon^3 - \frac{199}{128} \sqrt{2}\, k \zeta_0 \varepsilon^3$
η^2	$\frac{65}{384} k \varepsilon^3 + \frac{53}{576} k \zeta_0 \varepsilon^3$	$-\frac{5455}{27648} \sqrt{6}\, \varepsilon^3 + \frac{21943}{82944} \sqrt{6}\, \zeta_0 \varepsilon^3$
ξ^3	$\frac{165}{128} \varepsilon^3 - \frac{69}{64} \zeta_0 \varepsilon^3$	$\frac{495}{1024} \sqrt{6}\, k \varepsilon^3 - \frac{909}{1024} \sqrt{6}\, k \zeta_0 \varepsilon^3$
$\xi^2 \eta$	$-\frac{15}{64} \sqrt{3}\, k \varepsilon^3 + \frac{9}{32} \sqrt{3}\, k \zeta_0 \varepsilon^3$	$-\frac{505}{1024} \sqrt{2}\, \varepsilon^3 + \frac{791}{1024} \sqrt{2}\, \zeta_0 \varepsilon^3$
$\xi \eta^2$	$\frac{25}{128} \varepsilon^3 - \frac{5}{64} \zeta_0 \varepsilon^3$	$\frac{225}{1024} \sqrt{6}\, k \varepsilon^3 - \frac{315}{1024} \sqrt{6}\, k \zeta_0 \varepsilon^3$
η^3	0	$-\frac{375}{1024} \sqrt{2}\, \varepsilon^3 + \frac{225}{1024} \sqrt{2}\, \zeta_0 \varepsilon^3$

(15)

With $\zeta_0 = 0$ the scheme reduces to the scheme [M.N. (59)]. Introducing the values given in the scheme in the differential equations we find

$$\left.\begin{aligned} q_1 &= \frac{44785}{13824} \varepsilon^2 - \frac{48379}{10368} \zeta_0 \varepsilon^2, \\ q_2 &= \frac{7205}{4608} \sqrt{3}\, k \varepsilon^2 - \frac{3721}{1728} \sqrt{3}\, k \zeta_0 \varepsilon^2, \\ q_3 &= \frac{365}{1536} \sqrt{6}\, k \varepsilon^2 - \frac{6559}{4608} \sqrt{6}\, k \zeta_0 \varepsilon^2, \\ q_4 &= \frac{12835}{4608} \sqrt{2}\, \varepsilon^2 - \frac{23767}{4608} \sqrt{2}\, \zeta_0 \varepsilon^2. \end{aligned}\right\} \quad (16)$$

q_1, q_2, q_3 and q_4 must satisfy the two following conditions:—

$$\left.\begin{aligned}&\left(\omega^2+\frac{3}{4}\right)q_2+2\,\omega\,q_3-\frac{3}{4}\sqrt{3}\,k\,q_1 = 0,\\&\left(\omega^2-\frac{1}{2}\right)^2+\frac{27}{4}\,\delta-\left(\omega^2+\frac{9}{4}\right)q_1+\frac{3}{4}\sqrt{3}\,k\,q_2+2\,\omega\,q_4 = 0.\end{aligned}\right\}\quad(17)$$

Introducing the values from (16) in the first equation of condition we find that it is identically satisfied, thus obtaining a useful check on our calculations. The second equation of condition, which determines ω, now takes the form

$$\left(\omega^2-\frac{1}{2}\right)^2+\frac{27}{4}\,\delta-\frac{295}{864}\,\varepsilon^2+\frac{1867}{2592}\,\zeta_0\,\varepsilon^2 = 0. \qquad (18)$$

If for the sake of brevity we put

$$\omega^2-\frac{1}{2} = \zeta, \qquad (19)$$

we get upon subtraction of equation (6) from equation (18)

$$\zeta^2-\zeta_0^2+\frac{1867}{2592}\,\zeta_0\,\varepsilon^2 = 0. \qquad (20)$$

When ζ_0 differs from zero, this gives

$$\zeta = \zeta_0-\frac{1867}{5184}\,\varepsilon^2. \qquad (21)$$

From (19) and (21) we then find

$$\omega^2 = \frac{1}{2}+\zeta_0-\frac{1867}{5184}\,\varepsilon^2. \qquad (22)$$

We have now determined ω up to terms of the 2. order. This enables us to find the coefficients a_{-1} and b_{-1} from the two equations [M.N.(35)]. The calculation leads to the following expressions:—

$$\left.\begin{aligned} a_{-1} &= \left[\frac{3}{8} - \frac{3}{8}\zeta_0 - \frac{12571}{27648}\varepsilon^2\right]\sqrt{6}\,k\,\varepsilon, \\ b_{-1} &= \left[-\frac{5}{8} + \frac{1}{8}\zeta_0 + \frac{118231}{82944}\varepsilon^2\right]\sqrt{2}\,\varepsilon. \end{aligned}\right\} \qquad (23)$$

The 3. order coefficients a_3, a_{-3}, b_3 and b_{-3} are given by the expressions [M.N.(79)]:—

$$\left.\begin{aligned} a_3 &= \frac{7595}{55296}\varepsilon^3, & a_{-3} &= \frac{1405}{36864}\sqrt{6}\,k\,\varepsilon^3, \\ b_3 &= -\frac{1037}{18432}\sqrt{3}\,k\,\varepsilon^3, & b_{-3} &= \frac{10951}{110592}\sqrt{2}\,\varepsilon^3. \end{aligned}\right\} \qquad (24)$$

We have thus found the following series expressions for the co-ordinates of the infinitesimal periodic orbits around the libration point L_4, for values of the mass product close to the critical value $\frac{1}{27}$:—

$$\left.\begin{aligned} \xi = \left[-\frac{165}{128} + \frac{99}{64}\zeta_0\right]k\,\varepsilon^2 & + \varepsilon\cos\omega t \\ & + \left[\frac{3}{8} - \frac{3}{8}\zeta_0 - \frac{12571}{27648}\varepsilon^2\right]\sqrt{6}\,k\,\varepsilon\sin\omega t \\ & + \left[-\frac{11}{128} - \frac{1}{192}\zeta_0\right]k\,\varepsilon^2\cos 2\,\omega t \\ & + \left[-\frac{127}{864} + \frac{1381}{2592}\zeta_0\right]\sqrt{6}\,\varepsilon^2\sin 2\,\omega t \\ & + \frac{7595}{55296}\varepsilon^3\cos 3\,\omega t \\ & + \frac{1405}{36864}\sqrt{6}\,k\,\varepsilon^3\sin 3\,\omega t, \end{aligned}\right\} \qquad (25)$$

$$\begin{aligned}\eta = & \left[\frac{65}{384}-\frac{19}{192}\zeta_0\right]\sqrt{3}\,\varepsilon^2 \\ & +\left[-\frac{5}{8}+\frac{1}{8}\zeta_0+\frac{118231}{82944}\varepsilon^2\right]\sqrt{2}\,\varepsilon\sin\omega t \\ & +\left[\frac{79}{3456}+\frac{841}{5184}\zeta_0\right]\sqrt{3}\,\varepsilon^2\cos 2\,\omega t \\ & +\left[-\frac{13}{96}-\frac{29}{288}\zeta_0\right]\sqrt{2}\,k\,\varepsilon^2\sin 2\,\omega t \\ & -\frac{1037}{18432}\sqrt{3}\,k\,\varepsilon^3\cos 3\,\omega t \\ & +\frac{10951}{110592}\sqrt{2}\,\varepsilon^3\sin 3\,\omega t.\end{aligned}$$

Here k, ζ_0 and ω are determined by the equations

$$k=\sqrt{\frac{23}{27}}, \tag{26}$$

$$\zeta_0^2+\frac{27}{4}\delta-\frac{295}{864}\varepsilon^2=0, \tag{27}$$

$$\omega^2 = \frac{1}{2}+\zeta_0-\frac{1867}{5184}\varepsilon^2. \tag{28}$$

ε is the orbital constant, while δ is the excess of the mass product above the critical value.

In deriving ζ_0 from equation (27) we generally get two values, which are numerically equal and of opposite sign. It appears from equation (28) that the positive value of ζ_0 corresponds to short-period orbits, and the negative value to long-period orbits.

We can find the expression for the Jacobian constant from [M.N.(13)], inserting the values of ξ, η, ξ' and η' for $\omega t = 0$. The calculation gives

$$C = 2-\frac{5}{8}\zeta_0\varepsilon^2+\frac{2455}{41472}\varepsilon^4. \tag{29}$$

As a check the same calculation was made for $\omega t = \frac{\pi}{2}$. This leads to the same expression for C, as it should.

The two equations (27) and (28) determine $\omega^2 - \frac{1}{2}$, or ζ, as a function of δ and ε. A detailed investigation of this function now clears up completely the hitherto unsolved problems concerning the short-period and long-period orbital classes, for values of the mass product close to, or equal to, the critical value $\frac{1}{27}$.

The investigation is separated into three parts:—

1°. $\delta < 0$.

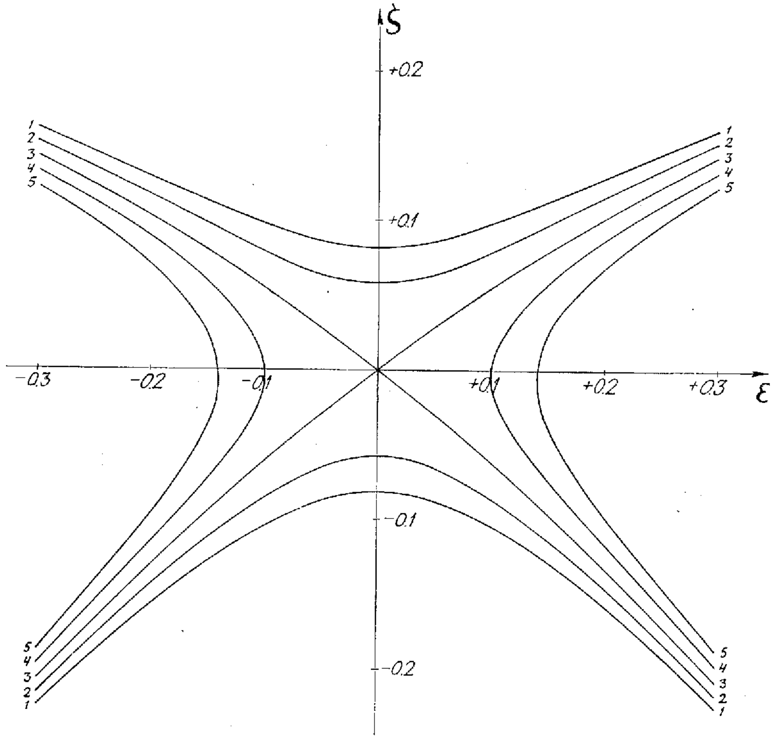
From equation (27) we find

$$\zeta_0 = \pm \sqrt{\frac{295}{864}\varepsilon^2 - \frac{27}{4}\delta}, \tag{30}$$

and then from equation (28)

$$\zeta = \pm \sqrt{\frac{295}{864}\varepsilon^2 - \frac{27}{4}\delta} \quad - \frac{1867}{5184}\varepsilon^2. \tag{31}$$

To any given value of ε there correspond two different values of ζ. The upper sign corresponds to short-period orbits, and the lower sign to long-period orbits. The Fig. shows a graphical representation of ζ as a function of ε. Curve 1 corresponds to $\delta = -0.0010$, curve 2 to $\delta = -0.0005$. The curve branches above the abscissa axis correspond to short-period orbits, those below to long-period orbits. The curves are symmetrical with regard to the ordinate axis, but unsymmetrical with regard to the abscissa axis. Two symmetrical points on the same curve branch correspond to identical orbits, since substitution



of $-\varepsilon$ for ε, and $\omega t+\pi$ for ωt, leaves the Fourier series for ξ and η unaffected.

2°. $\delta = 0$.

From equation (27) we get

$$\zeta_0 = \pm\sqrt{\frac{295}{864}}\,\varepsilon, \tag{32}$$

and then from equation (28)

$$\zeta = \pm\sqrt{\frac{295}{864}}\,\varepsilon - \frac{1867}{5184}\,\varepsilon^2. \tag{33}$$

The two parabolas that correspond to equation (33) are represented by curves 3 in the Fig. When ε has the sign in front of the square root, we get short-period orbits, with opposite sign we get long-period orbits.

3°. $\delta > 0$.

Real solutions of equation (27) only exist provided

$$\varepsilon^2 \geqq \frac{5832}{295} \delta. \tag{34}$$

Let ε_0 be the smallest value of the orbital constant for which (34) is satisfied. Then we get from equation (27)

$$\zeta_0 = \pm \sqrt{\frac{295}{864} (\varepsilon^2 - \varepsilon_0^2)}, \tag{35}$$

and finally from equation (28)

$$\zeta = \pm \sqrt{\frac{295}{864} (\varepsilon^2 - \varepsilon_0^2)} - \frac{1867}{5184} \varepsilon^2. \tag{36}$$

The upper sign corresponds to short-period orbits, the lower sign to long-period orbits. In the Fig. the curves 4 and 5 correspond to $\delta = 0.0005$, and $\delta = 0.0010$, respectively. Short-period and long-period orbits correspond to points on the same curve branch, and belong to the same orbital class. Curve branches symmetrical with respect to the ζ-axis correspond to the same orbital class, since substitution of $-\varepsilon$ for ε, and $\omega t + \pi$ for ωt, leaves the Fourier series for ξ and η unaffected.

When the mass product is greater than $\frac{1}{27}$ there exists an inner limiting orbit to the system of periodic orbits, corresponding to $\varepsilon = \varepsilon_0$. Since $\zeta_0 = 0$ for the limiting orbits, the Fourier series for ξ and η assume a particularly simple form. Comparison between these Fourier expressions and the Fourier expressions for the same orbit as given in [M.N.(87)] shows that there is, in two places, a small discrepancy, namely, a discrepancy in the terms of the 3. order in the coefficient of $\sin \omega t$ in the Fourier expression for ξ as well as in that for η.

Inside the inner limiting orbit no periodic orbits exist. However in this domain there exist asymptotic orbits, i. e. spiral orbits which, with an infinite number of windings, approach or recede from the libration point. As has been shown by E. STRÖMGREN[1], the shape of the individual spiral winding approaches the shape of the periodic orbits, as the value of the mass product approaches the critical value $\frac{1}{27}$.

[1] D. Kgl. Danske Vidensk. Selsk. Math.-fys. Medd. X, 11. 1930. — Publ. fra Københavns Obs. Nr. 70.

Indleveret til Selskabet den 5. Juni 1939.
Færdig fra Trykkeriet den 16. August 1939.