Det Kgl. Danske Videnskabernes Selskab.
Mathematisk-fysiske Meddelelser. **XVII,** 3.

NOTE
ON DIVIDED DIFFERENCES

BY

J. F. STEFFENSEN

KØBENHAVN
EJNAR MUNKSGAARD
1939

Printed in Denmark.
Bianco Lunos Bogtrykkeri A/S.

1. The number of general theorems concerning divided differences is so small that any addition to the list may, perhaps, be welcome. The unexpectedly simple theorem which forms the object of this Note seems, as far as I have been able to ascertain, to be new; it may be regarded as a generalization of LEIBNIZ' formula for the r^{th} derivate of a product of two functions.

The notation will be that of the author's book "Interpolation". Thus, for instance, $\varphi(x_0 x_1 \ldots x_r)$ will be the r^{th} divided difference of $\varphi(x)$, formed with the arguments x_0, $x_1, \ldots x_r$. In order to save space we shall, as a rule, only write the first and the last of the arguments, where no confusion is likely to arise.

Let, then,

$$\varphi(x) = f(x)\, g(x); \tag{1}$$

we propose to prove, by induction, that

$$\varphi(x_0 \ldots x_r) = \sum_{\nu=0}^{r} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_r). \tag{2}$$

It is readily ascertained that the formula is true for $r = 1$, that is,

$$\varphi(x_0 x_1) = f(x_0)\, g(x_0 x_1) + f(x_0 x_1)\, g(x_1).$$

We proceed to show that, if the formula is true for one value of r, it also holds for the following value.

In order to prove this, we employ the identity

$$\varphi(x_0 \ldots x_{r+1}) = \frac{\varphi(x_0 \ldots x_r) - \varphi(x_1 \ldots x_{r+1})}{x_0 - x_{r+1}}.$$

Applying this, we find, assuming (2) to be true for some particular value of r,

$$(x_0 - x_{r+1})\,\varphi(x_0 \ldots x_{r+1}) =$$

$$= \sum_{\nu=0}^{r} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_r) - \sum_{\nu=0}^{r} f(x_1 \ldots x_{\nu+1})\, g(x_{\nu+1} \ldots x_{r+1}).$$

Inserting, in this,

$$f(x_1 \ldots x_{\nu+1}) = f(x_0 \ldots x_\nu) - (x_0 - x_{\nu+1})\, f(x_0 \ldots x_{\nu+1}),$$

we find

$$(x_0 - x_{r+1})\,\varphi(x_0 \ldots x_{r+1}) =$$

$$= \sum_{\nu=0}^{r} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_r) - \sum_{\nu=0}^{r} f(x_0 \ldots x_\nu)\, g(x_{\nu+1} \ldots x_{r+1})$$

$$+ \sum_{\nu=0}^{r} (x_0 - x_{\nu+1})\, f(x_0 \ldots x_{\nu+1})\, g(x_{\nu+1} \ldots x_{r+1}).$$

In the second sum on the right we introduce

$$g(x_{\nu+1} \ldots x_{r+1}) = g(x_\nu \ldots x_r) - (x_\nu - x_{r+1})\, g(x_\nu \ldots x_{r+1}),$$

and in the third sum we write $\nu - 1$ instead of ν. Thus, we obtain

$$(x_0 - x_{r+1})\,\varphi(x_0 \ldots x_{r+1}) =$$

$$= \sum_{\nu=0}^{r} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_r) - \sum_{\nu=0}^{r} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_r)$$

$$+ \sum_{\nu=0}^{r} (x_\nu - x_{r+1})\, f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_{r+1})$$

$$+ \sum_{\nu=1}^{r+1} (x_0 - x_\nu)\, f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_{r+1})$$

which reduces to

$$(x_0 - x_{r+1}) \sum_{\nu=0}^{r+1} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_{r+1}),$$

so that

$$\varphi(x_0 \ldots x_{r+1}) = \sum_{\nu=0}^{r+1} f(x_0 \ldots x_\nu)\, g(x_\nu \ldots x_{r+1}).$$

But this is (2) with $r+1$ instead of r, so that (2) is true for all values of r.

2. Formula (2) contains several well-known formulas as particular cases. Thus, if we make all the arguments x_ν tend to the same point x, we obtain, if the derivates exist,

$$\frac{\varphi^{(r)}(x)}{r!} = \sum_{\nu=0}^{r} \frac{f^{(\nu)}(x)}{\nu!} \frac{g^{(r-\nu)}(x)}{(r-\nu)!}$$

which may also be written

$$D^r f(x)\, g(x) = \sum_{\nu=0}^{r} \binom{r}{\nu} D^\nu f(x) \cdot D^{r-\nu} g(x). \tag{3}$$

This is the theorem of Leibniz referred to above.

Putting next, in succession, $x_\nu = x + \nu$, $x_\nu = x - \nu$ and $x_\nu = x - \frac{r}{2} + \nu$ and making use of the relations

$$f(x,\ x+1, \ldots, x+n) = \frac{\triangle^n f(x)}{n!},$$

$$f(x,\ x-1, \ldots, x-n) = \frac{\nabla^n f(x)}{n!},$$

$$f\left(x - \frac{n}{2},\ x - \frac{n}{2} + 1, \ldots, x + \frac{n}{2}\right) = \frac{\delta^n f(x)}{n!},$$

we obtain, in analogy with (3), the three well-known relations

$$\triangle^r f(x)\,g(x) = \sum_{\nu=0}^{r} \binom{r}{\nu} \triangle^\nu f(x) \cdot \triangle^{r-\nu} g(x+\nu), \tag{4}$$

$$\nabla^r f(x)\,g(x) = \sum_{\nu=0}^{r} \binom{r}{\nu} \nabla^\nu f(x) \cdot \nabla^{r-\nu} g(x-\nu), \tag{5}$$

$$\delta^r f(x)\,g(x) = \sum_{\nu=0}^{r} \binom{r}{\nu} \delta^\nu f\left(x-\frac{r-\nu}{2}\right) \cdot \delta^{r-\nu} g\left(x+\frac{\nu}{2}\right). \tag{6}$$

3. We now put

$$f(x) = F(t) - F(x), \qquad g(x) = \frac{1}{t-x}, \tag{7}$$

so that

$$g(x_\nu \ldots x_r) = \frac{1}{(t-x_\nu)\ldots(t-x_r)}$$

and

$$\varphi(x) = \frac{F(t)-F(x)}{t-x}. \tag{8}$$

Inserting in (2), we obtain, keeping the first term on the right apart,

$$\varphi(x_0 \ldots x_r) = \frac{F(t)-F(x_0)}{(t-x_0)\ldots(t-x_r)} - \sum_{\nu=1}^{r} \frac{F(x_0 \ldots x_\nu)}{(t-x_\nu)\ldots(t-x_r)}$$

or, solving for $F(t)$,

$$F(t) = \sum_{\nu=0}^{r} (t-x_0)\ldots(t-x_{\nu-1}) F(x_0 \ldots x_\nu) + R, \tag{9}$$

$$R = (t-x_0)\ldots(t-x_r)\,\varphi(x_0 \ldots x_r), \tag{10}$$

where the factorial $(t-x_0)\ldots(t-x_{\nu-1})$ for $\nu = 0$ is interpreted as 1.

This is Newton's interpolation formula with divided differences and a remainder term differing slightly from the usual form. The latter is obtained by observing that, if we put

$$\theta_p f(x_0) = f(x_0 x_p), \qquad \theta f(x_0) = f(x_0 t), \tag{11}$$

θ_p and θ being symbols acting on x_0 alone, then, since $\varphi(x_0) = \theta F(x_0)$,

$$\varphi(x_0 \ldots x_r) = \theta_r \theta_{r-1} \ldots \theta_1 \varphi(x_0) = \theta_r \theta_{r-1} \ldots \theta_1 \theta F(x_0),$$

or

$$\varphi(x_0 \ldots x_r) = F(t x_0 \ldots x_r), \tag{12}$$

so that

$$R = (t-x_0) \ldots (t-x_r) F(t x_0 \ldots x_r). \tag{13}$$

But from (10) we obtain in particular cases forms of the remainder which are worth noting. Thus, for instance, if all the arguments tend to the same point x, we find TAYLOR's formula

$$F(t) = \sum_{\nu=0}^{r} \frac{(t-x)^{\nu}}{\nu!} F^{(\nu)}(x) + R \tag{14}$$

with the remainder

$$R = \frac{(t-x)^{r+1}}{r!} D^r \frac{F(t) - F(x)}{t-x}, \tag{15}$$

the operator D acting on x.

Further, putting $x_\nu = x + \nu$, (9) and (10) yield

$$F(t) = \sum_{\nu=0}^{r} \frac{(t-x)^{(\nu)}}{\nu!} \triangle^{\nu} F(x) + R, \tag{16}$$

$$R = \frac{(t-x)^{(r+1)}}{r!} \triangle^{r} \frac{F(t) - F(x)}{t-x}, \tag{17}$$

where $\triangle$ acts on x. This is the interpolation formula with descending differences and a remainder term which has already been given by BOOLE[1].

[1] Finite Differences, 3rd ed., p. 146.

Finally, putting $x_\nu = x-\nu$, we find the interpolation formula with ascending differences

$$F(t) = \sum_{\nu=0}^{r} \frac{(t-x)^{(-\nu)}}{\nu!} \nabla^\nu F(x) + R, \tag{18}$$

$$R = \frac{(t-x)^{(-r-1)}}{r!} \nabla^r \frac{F(t)-F(x)}{t-x}, \tag{19}$$

∇ acting on x.

It is evidently easy to transform the preceding remainder terms to the usual forms.

4. It is easy to extend the formula (2) to a product of any number of functions. Thus, if

$$f(x) = f_1(x) f_2(x), \qquad g(x) = f_3(x),$$

we have

$$f(x_0 \ldots x_\nu) = \sum_{\mu=0}^{\nu} f_1(x_0 \ldots x_\mu) f_2(x_\mu \ldots x_\nu),$$

and

$$\varphi(x) = f_1(x) f_2(x) f_3(x)$$

$$\varphi(x_0 \ldots x_r) =$$
$$= \sum_{\nu=0}^{r} \sum_{\mu=0}^{\nu} f_1(x_0 \ldots x_\mu) f_2(x_\mu \ldots x_\nu) f_3(x_\nu \ldots x_r).$$

Generally, if

$$\varphi(x) = f_1(x) f_2(x) \ldots f_n(x), \tag{20}$$

we may write

$$\varphi(x_0 \ldots x_r) =$$
$$= \sum f_1(x_0 \ldots x_\alpha) f_2(x_\alpha \ldots x_\beta) f_3(x_\beta \ldots x_\gamma) \ldots f_n(x_\varrho \ldots x_r), \tag{21}$$

the summation extending to all values of $\alpha, \beta, \gamma, \ldots \varrho$ for which

$$0 \leq \alpha \leq \beta \leq \gamma \leq \cdots \leq \varrho \leq r. \tag{22}$$

Thus, for instance, if $n=3$ we may at once write down

$$\begin{aligned}\varphi(x_0x_1x_2) &= f_1(x_0)f_2(x_0)f_3(x_0x_1x_2)\\ &+ f_1(x_0)f_2(x_0x_1)f_3(x_1x_2)\\ &+ f_1(x_0)f_2(x_0x_1x_2)f_3(x_2)\\ &+ f_1(x_0x_1)f_2(x_1)f_3(x_1x_2)\\ &+ f_1(x_0x_1)f_2(x_1x_2)f_3(x_2)\\ &+ f_1(x_0x_1x_2)f_2(x_2)f_3(x_2).\end{aligned}$$

If, in (21), we let all the arguments tend to the same point x, we get

$$\frac{\varphi^{(r)}(x)}{r!}=\sum\frac{f_1^{(\alpha)}(x)}{\alpha!}\,\frac{f_2^{(\beta-\alpha)}(x)}{(\beta-\alpha)!}\cdots\frac{f_n^{(r-\varrho)}(x)}{(r-\varrho)!}$$

and from this, putting $\alpha=\nu_1$, $\beta-\alpha=\nu_2,\ldots,r-\varrho=\nu_n$,

$$\varphi^{(r)}(x)=\sum\frac{r!}{\nu_1!\,\nu_2!\ldots\nu_n!}f_1^{(\nu_1)}(x)\,f_2^{(\nu_2)}(x)\ldots f_n^{(\nu_n)}(x), \quad (23)$$

the summation extending to all values of $\nu_1,\nu_2,\ldots\nu_n$ for which

$$\nu_1+\nu_2+\cdots+\nu_n=r. \quad (24)$$

This is the theorem of Leibniz for a product of n functions. It may be written symbolically in the form

$$\varphi^{(r)}=(f_1+f_2+\cdots+f_n)^r \quad (25)$$

with the convention that, after expanding, f^ν should be replaced by $f^{(\nu)}$. It should be noted that the zero powers of f cannot be omitted, since $f^{(0)}$ does not mean 1 but f.

If, in (21), we choose $x_\nu=x+\nu$, we find

$$\frac{\Delta^r\varphi(x)}{r!}=\sum\frac{\Delta^\alpha f_1(x)}{\alpha!}\,\frac{\Delta^{\beta-\alpha}f_2(x+\alpha)}{(\beta-\alpha)!}\cdots\frac{\Delta^{r-\varrho}f_n(x+\varrho)}{(r-\varrho)!}$$

or, in the notation (24),

$$\left.\begin{aligned} \triangle^r \varphi(x) &= \sum \frac{r!}{\nu_1!\,\nu_2!\dots\nu_n!}\times \\ \triangle^{\nu_1} f_1(x)\,\triangle^{\nu_2} f_2(x+\nu_1)\dots &\triangle^{\nu_n} f_n(x+\nu_1+\dots+\nu_{n-1}). \end{aligned}\right\}\quad(26)$$

Similarly, putting $x_\nu = x-\nu$, we obtain

$$\left.\begin{aligned} \nabla^r \varphi(x) &= \sum \frac{r!}{\nu_1!\,\nu_2!\dots\nu_n!}\times \\ \nabla^{\nu_1} f_1(x)\,\nabla^{\nu_2} f_2(x-\nu_1)\dots &\nabla^{\nu_n} f_n(x-\nu_1-\dots-\nu_{n-1}), \end{aligned}\right\}\quad(27)$$

and finally, making $x_\nu = x-\frac{r}{2}+\nu$,

$$\left.\begin{aligned} \delta^r \varphi(x) &= \sum \frac{r!}{\nu_1!\,\nu_2!\dots\nu_n!}\times \\ \delta^{\nu_1} f_1\left(x-\frac{r-\nu_1}{2}\right)\delta^{\nu_2} f_2&\left(x+\nu_1-\frac{r-\nu_2}{2}\right)\dots \\ \dots\,\delta^{\nu_n} f_n&\left(x+\nu_1+\dots+\nu_{n-1}-\frac{r-\nu_n}{2}\right). \end{aligned}\right\}\quad(28)$$

It is easy to put also (26), (27) and (28) into symbolic forms; but as these are more complicated than (25) and, therefore, not so useful, they seem hardly worth recording.

5. As an application of (21) we put

$$f_\nu(x) = \frac{1}{t-x}, \qquad \varphi(x) = \frac{1}{(t-x)^n}, \tag{29}$$

and obtain without difficulty

$$\left.\begin{aligned} &\varphi(x_0\dots x_r) = \\ &= \frac{1}{(t-x_0)\dots(t-x_r)}\sum \frac{1}{(t-x_\alpha)(t-x_\beta)\dots(t-x_\varrho)} \end{aligned}\right\}\quad(30)$$

the summation extending to the values of $\alpha, \beta, \dots \varrho$ satisfying (22).

But since the degree of the product

$$(t-x_\alpha)(t-x_\beta)\dots(t-x_\varrho)$$

is the same as the number of the quantities $\alpha, \beta, \dots \varrho$ which is $n-1$, (30) may also be written

$$\left.\begin{aligned}&\varphi(x_0\dots x_r) = \\ &= \frac{1}{(t-x_0)\dots(t-x_r)}\sum\frac{1}{(t-x_0)^{\mu_0}\dots(t-x_r)^{\mu_r}},\end{aligned}\right\}\quad(31)$$

the summation extending to all values of $\mu_0, \mu_1, \dots \mu_r$ for which

$$\mu_0+\mu_1+\dots+\mu_r = n-1. \qquad (32)$$

Instead of (31) and (32) we may evidently write

$$\varphi(x_0\dots x_r) = \sum\frac{1}{(t-x_0)^{\lambda_0}\dots(t-x_r)^{\lambda_r}} \qquad (33)$$

where

$$\lambda_0+\lambda_1+\dots+\lambda_r = n+r, \qquad \lambda_\nu \geq 1. \qquad (34)$$

It thus appears that $\varphi(x_0\dots x_r)$ is the coefficient of z^{n+r} in the development of

$$\left.\frac{\dfrac{z}{t-x_0}\cdot\dfrac{z}{t-x_1}\cdots\dfrac{z}{t-x_r}}{\left(1-\dfrac{z}{t-x_0}\right)\left(1-\dfrac{z}{t-x_1}\right)\cdots\left(1-\dfrac{z}{t-x_r}\right)}\right\}\quad(35)$$

or the coefficient of z^{n-1} in the development of

$$\frac{1}{(t-x_0-z)(t-x_1-z)\dots(t-x_r-z)}. \qquad (36)$$

The number of terms in (33) is obtained by putting $t=1$, $x_\nu = 0$ for all ν, and is therefore, according to (36), the coefficient of z^{n-1} in the development of $(1-z)^{-r-1}$, that is, $\binom{r+n-1}{n-1}$.

6. Lastly, we consider the case

$$f_\nu(x) = \frac{1}{t_\nu - x}, \qquad \varphi(x) = \frac{1}{(t_1 - x)\dots(t_n - x)}, \tag{37}$$

assuming all t_ν different. Here, an abbreviation of the notation becomes necessary, and we shall write

$$t^{\alpha\beta} = (t - x_\alpha)(t - x_{\alpha+1})\dots(t - x_\beta). \tag{38}$$

We obtain, then, from (21)

$$\varphi(x_0 \dots x_r) = \sum \frac{1}{t_1^{0\alpha} t_2^{\alpha\beta} \dots t_n^{\varrho r}}, \tag{39}$$

the summation extending as before to (22).

But we have also, for instance by Lagrange's interpolation formula,

$$\varphi(x) = \frac{1}{(t_1 - x)\dots(t_n - x)} = \sum_{\nu=1}^{n} \frac{1}{K_\nu (t_\nu - x)} \tag{40}$$

where

$$K_\nu = (t_1 - t_\nu)(t_2 - t_\nu)\dots(t_{\nu-1} - t_\nu)\cdot(t_{\nu+1} - t_\nu)\dots(t_n - t_\nu), \tag{41}$$

so that

$$\varphi(x_0 \dots x_r) = \sum_{\nu=1}^{n} \frac{1}{K_\nu t_\nu^{0r}}. \tag{42}$$

We therefore obtain, by comparison of (42) and (39), the identity

$$\sum \frac{1}{t_1^{0\alpha} t_2^{\alpha\beta} \dots t_n^{\varrho r}} = \sum_{\nu=1}^{n} \frac{1}{K_\nu t_\nu^{0r}}. \tag{43}$$

In the particular case where $n = 2$ this becomes

$$\sum_{\nu=0}^{r} \frac{1}{t_1^{0\nu} t_2^{\nu r}} = \frac{1}{t_2 - t_1}\left(\frac{1}{t_1^{0r}} - \frac{1}{t_2^{0r}}\right). \tag{44}$$

Indleveret til Selskabet den 31. Marts 1939.
Færdig fra Trykkeriet den 22. Juni 1939.