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ON A GENERALIZATION OF NÖRLUND'S POLYNOMIALS

BY

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1. The class of polynomials considered in this paper will be denoted by $x^{\nu}_{\omega n}$ where x is the variable, ν the degree of the polynomial, while ω is called the range, n the order of the polynomial. The range may be any number, but the order is assumed to be integral (positive, negative or zero).

Writing

$$\underset{\omega}{\triangle} f(x) = \frac{f(x+\omega)-f(x)}{\omega},$$

so that $\underset{1}{\triangle} = \triangle$, we proceed to show that the polynomial $x^{\nu}_{\omega n}$ is completely determined, if we require that it shall satisfy the equations

$$\triangle x^{\nu}_{\omega n} = \nu x^{\nu-1}_{\omega,\, n-1}, \tag{1}$$

$$\underset{\omega}{\triangle} x^{\nu}_{\omega n} = \nu x^{\nu-1}_{\omega n}, \tag{2}$$

and the initial conditions

$$x^{0}_{\omega n} = 1, \qquad x^{\nu}_{\omega 0} = x(x-\omega)\cdots(x-\nu\omega+\omega). \tag{3}$$

It is obvious that a polynomial with such simple properties must have many applications in the theory of finite differences.

In order to prove the existence of the polynomial we will show that it can be effectively formed, and that the determination of its constants is unambiguous.

If the polynomial $x^{\nu}_{\omega n}$ exists, $(x+y)^{\nu}_{\omega n}$ will be a polynomial in y which can be developed in descending factorials $x^{(\nu)} \equiv x^{\nu}_{10}$

$$(x+y)^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \frac{y^{(s)}}{s!} \triangle^{s} x^{\nu}_{\omega n},$$

and this development is unique. But by (1)

$$\triangle^{s} x^{\nu}_{\omega n} = \nu^{(s)} x^{\nu-s}_{\omega,\, n-s}, \tag{4}$$

so that

$$(x+y)^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} y^{(s)} x^{\nu-s}_{\omega,\, n-s}. \tag{5}$$

Putting $y = \omega$, we obtain by (2), writing $\nu+1$ for ν,

$$x^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} \frac{(\omega-1)^{(s)}}{s+1} x^{\nu-s}_{\omega,\, n-s-1} \tag{6}$$

or

$$x^{\nu}_{\omega n} - x^{\nu}_{\omega,\, n-1} = \sum_{s=1}^{\nu} \binom{\nu}{s} \frac{(\omega-1)^{(s)}}{s+1} x^{\nu-s}_{\omega,\, n-s-1}. \tag{7}$$

On the right-hand side only polynomials of degree $< \nu$ occur. If all these are known, the polynomials of degree ν can be calculated in succession for every positive and negative order, $x^{\nu}_{\omega 0}$ being known. For $\nu = 1$, (7) reduces to

$$x^{1}_{\omega n} - x^{1}_{\omega,\, n-1} = \frac{\omega-1}{2} \tag{8}$$

which serves as starting-point for the calculation.

2. In practice we shall not use this mode of calculation which has only served to prove the existence of the polynomials. Before proceeding to their actual calculation we will develop some of their properties, and begin by showing that they contain as particular cases several important polynomials which have already been introduced into mathematical analysis.

Besides the obvious relations

$$x^{\nu}_{00} = x^{\nu}, \quad x^{\nu}_{10} = x^{(\nu)}, \tag{9}$$

obtained directly from (3), we note

$$x^{\nu}_{1n} = x^{(\nu)} \tag{10}$$

which is obtained from (2), putting $\omega = 1$ and comparing with (1). This comparison shows that $x^{\nu-1}_{1n}$ and $x^{\nu-1}_{1,n-1}$ have the same value which, therefore, is independent of n and, according to (9), equal to $x^{(\nu-1)}$.

Let us, next, examine what becomes of (2) as $\omega \to 0$. From (7) and (8) follows that $x^{\nu}_{\omega n}$ is a polynomial in x and ω. If D is the symbol of differentiation, we therefore have

$$(x+\omega)^{\nu}_{\omega n} = x^{\nu}_{\omega n} + \omega D x^{\nu}_{\omega n} + \frac{\omega^2}{2!} D^2 x^{\nu}_{\omega n} + \cdots,$$

the number of terms on the right being finite. Consequently

$$\underset{\omega}{\triangle}\, x^{\nu}_{\omega n} = D x^{\nu}_{\omega n} + \frac{\omega}{2!} D^2 x^{\nu}_{\omega n} + \cdots$$

and

$$\lim_{\omega \to 0} \underset{\omega}{\triangle}\, x^{\nu}_{\omega n} = D x^{\nu}_{0n}. \tag{11}$$

We see, then, that x^{ν}_{0n} is the polynomial, satisfying the equations

$$\left.\begin{aligned} \triangle\, x^{\nu}_{0n} &= \nu x^{\nu-1}_{0,n-1}, \\ D x^{\nu}_{0n} &= \nu x^{\nu-1}_{0n}, \end{aligned}\right\} \tag{12}$$

besides the initial conditions

$$x^{0}_{0n} = 1, \quad x^{\nu}_{00} = x^{\nu}. \tag{13}$$

This polynomial is, therefore, identical with Nörlund's generalization of Bernoulli's polynomial (for "zusammenfallende Spannen"), as (12) and (13) express properties of Nörlund's polynomial[1] and, as we have seen, are sufficient for determining it. That is, in Nörlund's notation,

$$x^{\nu}_{0n} = B^{(n)}_{\nu}(x). \tag{14}$$

It follows that x^{ν}_{01} is Bernoulli's polynomial, or

$$x^{\nu}_{01} = B_{\nu}(x) \tag{15}$$

whence, for Bernoulli's numbers,

$$0^{\nu}_{01} = B_{\nu}, \quad 0^{\nu}_{0n} = B^{(n)}_{\nu}. \tag{16}$$

3. The polynomials $x^{\nu}_{\omega n}$ may also be defined by means of a generating function which may for some purposes be found preferable. They may, in fact, be defined by the expansion

$$\left(\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}\right)^{n}(1+\omega t)^{\frac{x}{\omega}} = \sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\nu!}\,x^{\nu}_{\omega n}, \tag{17}$$

valid for sufficiently small values of $|t|$. For, performing the operation $\triangle$ on both sides of (17), we find

$$\left(\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}\right)^{n-1}(1+\omega t)^{\frac{x}{\omega}} = \sum_{\nu=0}^{\infty}\frac{t^{\nu-1}}{\nu!}\triangle\, x^{\nu}_{\omega n}$$

and, comparing with (17), writing $n-1$ for n, (1) results. Similarly, if we perform the operation $\underset{\omega}{\triangle}$ on (17), we have

$$\left(\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1}\right)^{n}(1+\omega t)^{\frac{x}{\omega}} = \sum_{\nu=0}^{\infty}\frac{t^{\nu-1}}{\nu!}\underset{\omega}{\triangle}\, x^{\nu}_{\omega n},$$

and comparison with (17) leads to (2). Finally, the initial conditions (3) are obtained from (17) for $t = 0$ and for $n = 0$ respectively, as

[1] Nörlund: Differenzenrechnung, p. 130—1; Mémoire sur les polynomes de Bernoulli, Acta mathematica 43 (1920) p. 121.

$$(1+\omega t)^{\frac{x}{\omega}} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}\, x(x-\omega)\cdots(x-\nu\omega+\omega).$$

If $x = 0$, $n = 1$, (17) becomes

$$\frac{t}{(1+\omega t)^{\frac{1}{\omega}}-1} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}\, 0^\nu_{\omega 1}\,. \tag{18}$$

The quantities $0^\nu_{\omega 1}$ are well known; they are polynomials in ω, called Lubbock's polynomials. They are usually introduced by means of the expansion[1]

$$\frac{z}{(1+z)^{\frac{1}{\omega}}-1} = \sum_{\nu=0}^{\infty} z^\nu A_\nu. \tag{19}$$

As (19) is obtained from (18) by putting $t = \frac{z}{\omega}$, we have

$$0^\nu_{\omega 1} = \omega^{\nu-1}\,\nu!\,A_\nu. \tag{20}$$

Lubbock's polynomials as far as $0^7_{\omega 1}$ are:

$$\left.\begin{aligned}
0^1_{\omega 1} &= \frac{\omega-1}{2} \\
0^2_{\omega 1} &= -\frac{\omega^2-1}{6} \\
0^3_{\omega 1} &= \frac{\omega^2-1}{4}\,\omega \\
0^4_{\omega 1} &= -\frac{\omega^2-1}{30}(19\,\omega^2-1) \\
0^5_{\omega 1} &= \frac{\omega^2-1}{4}\,\omega\,(9\,\omega^2-1) \\
0^6_{\omega 1} &= -\frac{\omega^2-1}{84}(863\,\omega^4-145\,\omega^2+2) \\
0^7_{\omega 1} &= \frac{\omega^2-1}{24}\,5\,\omega\,(275\,\omega^4-61\,\omega^2+2)
\end{aligned}\right\} \tag{21}$$

[1] See, for instance, Steffensen: "Interpolationslære" p. 140—1, where m is written for ω. Numerical values of A_ν are given in Institute of Actuaries' Text-Book II p. 471.

4. We shall now occupy ourselves with the practical calculation of the polynomials $x^{\nu}_{\omega n}$.

If, in (5), we put $x = 0$ and thereafter $y = x$, we have

$$x^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega, n-s}\, x^{(s)}, \tag{22}$$

so that our polynomials may be developed in descending factorials $x^{(s)}$, if the values of the polynomials for $x = 0$ are known.

But these values also appear in the development in factorials of the form $x^{s}_{\omega 0}$. For, if $P(x)$ is a polynomial of degree ν, we have

$$P(x) = \sum_{s=0}^{\nu} \frac{x^{s}_{\omega 0}}{s!} \underset{\omega}{\triangle}^{s} P(0); \tag{23}$$

hence

$$(x+y)^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \frac{y^{s}_{\omega 0}}{s!} \underset{\omega}{\triangle}^{s} x^{\nu}_{\omega n}.$$

But by (2)

$$\underset{\omega}{\triangle}^{s} x^{\nu}_{\omega n} = \nu^{(s)} x^{\nu-s}_{\omega n}, \tag{24}$$

so that

$$(x+y)^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} y^{s}_{\omega 0}\, x^{\nu-s}_{\omega n}. \tag{25}$$

This formula which is an analogon to (5) may be written symbolically

$$(x+y)^{\nu}_{\omega n} = (y_{\omega 0} + x_{\omega n})^{\nu} \tag{26}$$

where the right-hand member is to be developed by the binomial theorem.

If, in (25), we put $x = 0$ and then $y = x$, we have

$$x^{\nu}_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega n}\, x^{s}_{\omega 0} \tag{27}$$

or symbolically

$$x^{\nu}_{\omega n} = (x_{\omega 0} + 0_{\omega n})^{\nu}. \tag{28}$$

A comparison of (27) and (22) shows that if the values of $0^{\nu}_{\omega n}$ are known, we obtain simultaneously the developments of $x^{\nu}_{\omega n}$ in factorials $x^{(s)}$ and in factorials $x^{s}_{\omega 0}$. It seems, therefore, practical to calculate a table of the $0^{\nu}_{\omega n}$ which are a generalization of Lubbock's polynomials. This will be done presently.

5. If we differentiate (17) with respect to t and put, for abbreviation,

$$A = \frac{t}{(1+\omega t)^{\frac{1}{\omega}} - 1}, \qquad B = 1 + \omega t,$$

we obtain, as $\frac{dA}{dt} = \frac{A}{t}\left(1 - A B^{\frac{1}{\omega}-1}\right)$,

$$x A^n B^{\frac{x}{\omega}-1} + \frac{n}{t} A^n B^{\frac{x}{\omega}} - \frac{n}{t} A^{n+1} B^{\frac{x+1}{\omega}-1} = \sum_{\nu=1}^{\infty} \frac{t^{\nu-1}}{(\nu-1)!} x^{\nu}_{\omega n}$$

or, multiplying by Bt,

$$x t A^n B^{\frac{x}{\omega}} + n A^n B^{\frac{x+\omega}{\omega}} - n A^{n+1} B^{\frac{x+1}{\omega}} = (1+\omega t) \sum_{\nu=1}^{\infty} \frac{t^{\nu}}{(\nu-1)!} x^{\nu}_{\omega n}.$$

Now, by (17), (2) and (1),

$$A^n B^{\frac{x}{\omega}} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} x^{\nu}_{\omega n},$$

$$A^n B^{\frac{x+\omega}{\omega}} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} (x+\omega)^{\nu}_{\omega n}$$

$$= \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\nu!} \left(x^{\nu}_{\omega n} + \nu \omega x^{\nu-1}_{\omega n}\right),$$

$$A^{n+1} B^{\frac{x+1}{\omega}} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} (x+1)^\nu_{\omega, n+1}$$

$$= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \left(x^\nu_{\omega, n+1} + \nu x^{\nu-1}_{\omega n} \right).$$

Inserting these developments in the above equation and examining the coefficients of t^ν, we find

$$x^\nu_{\omega, n+1} = \left(1 - \frac{\nu}{n}\right) x^\nu_{\omega n} + \left[\left(1 - \frac{\nu - 1}{n}\right)\omega + \frac{x}{n} - 1\right] \nu x^{\nu-1}_{\omega n}. \quad (29)$$

We combine this formula with the formula obtained from (25) by putting $y = 1$ and making use of (1), that is

$$\nu x^{\nu-1}_{\omega, n-1} = \sum_{s=1}^{\nu} \binom{\nu}{s} 1^s_{\omega 0}\, x^{\nu-s}_{\omega n}$$

or, writing $\nu+1$, $n+1$ and $s+1$ for ν, n and s,

$$x^\nu_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} \frac{1^{s+1}_{\omega 0}}{s+1} x^{\nu-s}_{\omega, n+1}.$$

If, in (29), we write $\nu - s$ for ν, we have

$$x^{\nu-s}_{\omega, n+1} = \left(1 - \frac{\nu - s}{n}\right) x^{\nu-s}_{\omega n} + \left[\left(1 - \frac{\nu - s - 1}{n}\right)\omega + \frac{x}{n} - 1\right](\nu - s) x^{\nu-s-1}_{\omega n},$$

and, inserting this expression in the preceding equation, we find after reduction, replacing ν by $\nu+1$,

$$x^{\nu+1}_{\omega n} = \sum_{s=0}^{\nu} \binom{\nu}{s} \left[\frac{n - \nu + s}{s+2}(\omega + 1) + x - n\right] \frac{1^{s+1}_{\omega 0}}{s+1} x^{\nu-s}_{\omega n}. \quad (30)$$

Putting in succession $\nu = 0, 1, 2, \ldots$ we obtain all the polynomials $x^\nu_{\omega n}$.

6. Instead of calculating these directly, we decided to calculate the $0^\nu_{\omega n}$ which are obtained by putting $x = 0$ in

(30). Even the expressions of the $0^{\nu}_{\omega n}$ become rapidly complicated, but the first few of them may be written comparatively simply by introducing the notation

$$\alpha = \frac{n}{2}(\omega - 1) \tag{31}$$

and developing in factorials $\alpha^{s}_{\omega 0}$. Eliminating n by this relation, putting $x = 0$ and keeping the first term on the right by itself, we obtain from (30)

$$0^{\nu+1}_{\omega n} = \left[\alpha - \frac{\nu}{2}(\omega+1)\right] 0^{\nu}_{\omega n}$$
$$+ \sum_{s=1}^{\nu} \binom{\nu}{s} \frac{2\alpha(s+1-\omega) - (\nu-s)(1-\omega^2)}{(s+1)(s+2)} (1-2\omega)^{s-1}_{\omega 0}\, 0^{\nu-s}_{\omega n}. \tag{32}$$

By this formula we find in succession:

$$0^{0}_{\omega n} = 1$$

$$0^{1}_{\omega n} = \alpha$$

$$0^{2}_{\omega n} = \alpha^{2}_{\omega 0} + \frac{\omega+1}{6}\alpha$$

$$0^{3}_{\omega n} = \alpha^{3}_{\omega 0} + \frac{\omega+1}{2}\alpha^{2}_{\omega 0}$$

$$0^{4}_{\omega n} = \alpha^{4}_{\omega 0} + (\omega+1)\alpha^{3}_{\omega 0} + \frac{(\omega+1)^2}{12}\alpha^{2}_{\omega 0} - \frac{\omega+1}{60}(1-2\omega)^{2}_{\omega 0}\,\alpha$$

$$0^{5}_{\omega n} = \alpha^{5}_{\omega 0} + 5\frac{\omega+1}{3}\alpha^{4}_{\omega 0} + 5\frac{(\omega+1)^2}{12}\alpha^{3}_{\omega 0} - \frac{\omega+1}{12}(1-2\omega)^{2}_{\omega 0}\,\alpha^{2}_{2\omega, 0}$$

$$\left.\begin{aligned} 0^{6}_{\omega n} = \alpha^{6}_{\omega 0} + 5\frac{\omega+1}{2}\alpha^{5}_{\omega 0} + 5\frac{(\omega+1)^2}{4}\alpha^{4}_{\omega 0} - \frac{\omega+1}{72}(103\,\omega^2 - 100\,\omega + 13)\,\alpha^{3}_{\omega 0} \\ + \frac{\omega+1}{504}(1-2\omega)^{2}_{\omega 0}\left[21(11\omega-1)\,\alpha^{2}_{2\omega, 0} + 2(\omega-2)(5\omega-1)\,\alpha\right] \end{aligned}\right\}$$

$$\left.\begin{aligned} 0^{7}_{\omega n} = \alpha^{7}_{\omega 0} + 7\frac{\omega+1}{2}\alpha^{6}_{\omega 0} + 35\frac{(\omega+1)^2}{12}\alpha^{5}_{\omega 0} - 7\frac{\omega+1}{72}(31\omega^2 - 40\,\omega + 1)\,\alpha^{4}_{\omega 0} \\ + \frac{\omega+1}{72}(5\omega-1)(1-2\omega)^{2}_{\omega 0}\left[21\,\alpha^{3}_{\omega 0} - 4(10\,\omega+1)\,\alpha^{2}_{2\omega, 0} - 2\omega(\omega-2)\,\alpha\right] \end{aligned}\right\}$$

It should be noted that $\alpha^2_{\omega 0}$ has, in $0^5_{\omega n}$, $0^6_{\omega n}$ and $0^7_{\omega n}$, been replaced by $\alpha^2_{2\omega, 0}$ by means of the obvious relation

$$\alpha^2_{\omega 0} = \alpha^2_{2\omega, 0} + \omega\alpha \tag{33}$$

which simplifies the writing.

7. Without attempting completeness, we proceed to derive a few of the simplest properties of the polynomials $x^\nu_{\omega n}$.

If, in (17), instead of x, ω and t we write respectively $n-x$, $-\omega$, and $-t$, we find

$$x^\nu_{\omega n} = (-1)^\nu (n-x)^\nu_{-\omega, n}. \tag{34}$$

In particular we have, for $x = n$,

$$n^\nu_{\omega n} = (-1)^\nu\, 0^\nu_{-\omega, n} \tag{35}$$

and for $x = \frac{n}{2}$

$$\left(\frac{n}{2}\right)^\nu_{\omega n} = (-1)^\nu \left(\frac{n}{2}\right)^\nu_{-\omega, n}. \tag{36}$$

It follows that $\left(\frac{n}{2}\right)^{2\nu}_{\omega n}$ contains only even powers of ω, $\left(\frac{n}{2}\right)^{2\nu+1}_{\omega n}$ only odd powers of ω.

If $\omega = -1$, we obtain from (34) by (10)

$$x^\nu_{-1, n} = (x+\nu-n-1)^{(\nu)}. \tag{37}$$

An important particular case is found from (29). Putting $n = \nu$ we have first

$$x^\nu_{\omega, \nu+1} = (x+\omega-\nu)\, x^{\nu-1}_{\omega\nu}$$

and by repeated application of this formula

$$x^\nu_{\omega, \nu+1} = (x+\omega-1)^{(\nu)}. \tag{38}$$

The polynomials of negative order ($n = -s$) serve for expressing the differences of $x^\nu_{\omega 0}$. Putting $n = 0$ in (4) we have immediately

$$\underset{\omega}{\triangle^s} x^\nu_{\omega 0} = \nu^{(s)} x^{\nu-s}_{\omega, -s}. \tag{39}$$

The differences of x^ν are obtained from this formula by putting $\omega = 0$, that is

$$\triangle^s x^\nu = \nu^{(s)} x^{\nu-s}_{0, -s} \tag{40}$$

whence for the "differences of nothing"

$$\triangle^s 0^\nu = \nu^{(s)} 0^{\nu-s}_{0, -s}. \tag{41}$$

The differences $\underset{\omega}{\triangle^s} x^{(\nu)}$ are obtained as follows. If, in (38), we replace x by $x - \omega + 1$, we have

$$x^{(\nu)} = (x - \omega + 1)^\nu_{\omega, \nu+1}$$

whence

$$\underset{\omega}{\triangle^s} x^{(\nu)} = \nu^{(s)} (x - \omega + 1)^{\nu-s}_{\omega, \nu+1}. \tag{42}$$

In particular, we have for $\omega = 0$

$$D^s x^{(\nu)} = \nu^{(s)} (x + 1)^{\nu-s}_{0, \nu+1}. \tag{43}$$

We need hardly repeat that (40), (41) and (43) as well as all other formulas obtained by putting $\omega = 0$, are, owing to (14), identical with those found by Nörlund. We shall therefore, as a rule, omit the results obtained for $\omega = 0$.

8. Our polynomials facilitate several elementary developments, such as $x^\nu_{\omega 0}$ in polynomials $x^{(s)}$, $x^{(\nu)}$ in polynomials $x^s_{\omega 0}$, etc. For instance, we find immediately from (22), putting $n = 0$,

$$x^\nu_{\omega 0} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega, -s} x^{(s)}. \tag{44}$$

From this, the development of $x^{(\nu)}$ in polynomials $x^s_{\omega 0}$ is obtained by substitution. For, putting $x = \omega y$, we have

$$\omega^\nu y^{(\nu)} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega, -s} (\omega y)^{(s)}$$

and hence, replacing y by x and writing $\frac{1}{\omega}$ for ω,

$$x^{(\nu)} = \sum_{s=0}^{\nu} \binom{\nu}{s} \omega^{\nu-s} \, 0^{\nu-s}_{\frac{1}{\omega}, -s} \, x^s_{\omega 0}. \tag{45}$$

A development which will be useful later on, is the development of $x^\nu_{\omega 1}$ in polynomials $x^s_{\omega 0}$. It is obtained from (25), putting $x = 0$, $n = 1$ and replacing thereafter y by x. The result is

$$x^\nu_{\omega 1} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^{\nu-s}_{\omega 1} \, x^s_{\omega 0}. \tag{46}$$

Another useful special result is the development

$$(x+y)^\nu_{\omega 0} = \sum_{s=0}^{\nu} \binom{\nu}{s} y^{(s)} x^{\nu-s}_{\omega, -s} \tag{47}$$

which is obtained from (5) for $n = 0$; it contains, of course, (44).

It may finally be mentioned that the differential coefficient of $x^\nu_{\omega n}$ may in several ways be expressed as a linear function of these polynomials. If, in (5) and (25), the first term on the right is transferred to the left, we obtain by dividing by y and letting $y \to 0$

$$Dx^\nu_{\omega n} = \sum_{s=1}^{\nu} \frac{(-1)^{s-1}}{s} \nu^{(s)} x^{\nu-s}_{\omega, n-s}, \tag{48}$$

$$Dx^\nu_{\omega n} = \sum_{s=1}^{\nu} \frac{(-1)^{s-1}}{s} \omega^{s-1} \nu^{(s)} x^{\nu-s}_{\omega n}. \tag{49}$$

9. Several important summation-problems can be solved by the polynomials $x^\nu_{\omega n}$, first of all the summation of the polynomials themselves. If, in (1), instead of x, ν and n we put $x+s$, $\nu+1$ and $n+1$ respectively, we find by summation from $s = 0$ to $s = k-1$

$$\sum_{s=0}^{k-1} (x+s)^{\nu}_{\omega n} = \frac{1}{\nu+1}\left[(x+k)^{\nu+1}_{\omega,\, n+1} - x^{\nu+1}_{\omega,\, n+1}\right]. \tag{50}$$

In particular we have, for $x=0$, $n=0$,

$$\sum_{s=0}^{k-1} s^{\nu}_{\omega 0} = \frac{1}{\nu+1}\left(k^{\nu+1}_{\omega 1} - 0^{\nu+1}_{\omega 1}\right); \tag{51}$$

the polynomials $x^{\nu}_{\omega 1}$ of which use is made on the right have already been given by (46).

From (2) we obtain, replacing x and ν by $x+s\omega$ and $\nu+1$, by summation from $s=0$ to $s=k-1$

$$\sum_{s=0}^{k-1} (x+s\omega)^{\nu}_{\omega n} = \frac{1}{(\nu+1)\,\omega}\left[(x+k\omega)^{\nu+1}_{\omega n} - x^{\nu+1}_{\omega n}\right]. \tag{52}$$

If, in particular, $\omega = \frac{1}{k}$, we have, owing to (1),

$$\frac{1}{k}\sum_{s=0}^{k-1}\left(x+\frac{s}{k}\right)^{\nu}_{\frac{1}{k},\, n} = x^{\nu}_{\frac{1}{k},\, n-1} \tag{53}$$

whence for $n=1$

$$\frac{1}{k}\sum_{s=0}^{k-1}\left(x+\frac{s}{k}\right)^{\nu}_{\frac{1}{k},\, 1} = x^{\nu}_{\frac{1}{k},\, 0}. \tag{54}$$

In (53) we may let $k \to \infty$, the result being Nörlund's »Spannenintegral«

$$\int_x^{x+1} x^{\nu}_{0n}\, dx = x^{\nu}_{0,\, n-1}. \tag{55}$$

10. Our polynomials lend themselves conveniently to treatment by the so-called »symbolic« methods. I assume that the reader is familiar with the conditions under which such treatment is legitimate,[1] the main point being that if $f(x)$ can be expanded in powers of x, and if $P(x)$ is a polynomial, then $f(\triangle)\,P(x)$ has a meaning and signi-

[1] See, for instance, "Interpolationslære" § 18.

fies the result of expanding $f(\triangle)$ in powers of $\triangle$ and applying this composite operation to $P(x)$ whereby all terms beyond a certain order vanish.

As an example, let us consider the operations $\underset{\omega}{\triangle}^{r} \triangle^{-r}$ and $\triangle^{r} \underset{\omega}{\triangle}^{-r}$ where r is integral and ≥ 0. These operations are unambiguous, as the indetermination of $\triangle^{-r}$ and $\underset{\omega}{\triangle}^{-r}$ is cancelled by $\underset{\omega}{\triangle}^{r}$ and $\triangle^{r}$ respectively. Therefore, if $P(x)$ is a polynomial, the polynomial $Q(x)$

$$Q(x) = \underset{\omega}{\triangle}^{r} \triangle^{-r} P(x) \tag{56}$$

is perfectly determined, and we have, as is seen by performing the operation $\triangle^{r} \underset{\omega}{\triangle}^{-r}$ on both sides,

$$P(x) = \triangle^{r} \underset{\omega}{\triangle}^{-r} Q(x). \tag{57}$$

Now, if the symbol E is defined by

$$E^{y} f(x) = f(x+y),$$

we have

$$\triangle = E - 1, \qquad \underset{\omega}{\triangle} = \frac{E^{\omega} - 1}{\omega},$$

$$E = \left(1 + \omega \underset{\omega}{\triangle}\right)^{\frac{1}{\omega}},$$

$$\triangle = \left(1 + \omega \underset{\omega}{\triangle}\right)^{\frac{1}{\omega}} - 1.$$

If, in (56), we replace x by $x+y$, we therefore have

$$Q(x+y) = \underset{\omega}{\triangle}^{r} \triangle^{-r} E^{y} P(x)$$

$$= \left(\frac{\underset{\omega}{\triangle}}{\left(1+\omega \underset{\omega}{\triangle}\right)^{\frac{1}{\omega}} - 1}\right)^{r} \left(1 + \omega \underset{\omega}{\triangle}\right)^{\frac{y}{\omega}} P(x)$$

or, comparing with (17),

$$Q(x+y) = \sum_{s=0}^{\infty} \frac{y^{s}_{\omega r}}{s!} \underset{\omega}{\triangle}^{s} P(x). \tag{58}$$

The series on the right contains, of course, only a finite number of terms, but it is convenient to retain ∞ as the upper limit of summation instead of specifying the degree of the polynomial $P(x)$.

If, in (58), we introduce the expression (57) for $P(x)$, we have

$$Q(x+y) = \sum_{s=0}^{\infty} \frac{y^s_{\omega r}}{s!} \triangle^r \underset{\omega}{\triangle}^{s-r} Q(x). \tag{59}$$

This is the expansion of an arbitrary polynomial in polynomials $y^s_{\omega r}$. If, on both sides of (59), we perform the operation $\underset{\omega}{\triangle}^r$ we obtain the following formula which may be called the generalized Euler-Maclaurin formula for a polynomial

$$\underset{\omega}{\triangle}^r Q(x+y) = \sum_{s=0}^{\infty} \frac{y^s_{\omega r}}{s!} \underset{\omega}{\triangle}^s \triangle^r Q(x). \tag{60}$$

The extention of these formulas to other functions than polynomials is an important problem with which I hope to occupy myself on another occasion.

11. As an application of (60) we put $Q(x) = x^{\nu+r}_{\omega,\, n+r}$ and find

$$(x+y)^{\nu}_{\omega,\, n+r} = \sum_{s=0}^{\nu} \binom{\nu}{s} y^s_{\omega r}\, x^{\nu-s}_{\omega n} \tag{61}$$

or symbolically

$$(x+y)^{\nu}_{\omega,\, n+r} = (y_{\omega r} + x_{\omega n})^{\nu}. \tag{62}$$

This is a generalization of (25) or (26) which are obtained for $r = 0$. In particular, we have for $y = 0$, $r = 1$

$$x^{\nu}_{\omega,\, n+1} = \sum_{s=0}^{\nu} \binom{\nu}{s} 0^s_{\omega 1}\, x^{\nu-s}_{\omega n}. \tag{63}$$

Eliminating $x^{\nu}_{\omega,\, n+1}$ between this equation and (29) we find by (31)

$$x^{\nu}_{\omega n} = \left[\alpha - (\nu-1)\,\omega + x\right] x^{\nu-1}_{\omega n} - 2\,\alpha\,\frac{\omega+1}{\nu} \sum_{s=2}^{\nu} \binom{\nu}{s} \frac{0^{s}_{\omega 1}}{\omega^2 - 1}\, x^{\nu-s}_{\omega n} \qquad (64)$$

which, $0^{s}_{\omega 1}$ being known by (21), may serve for the successive calculation of the $x^{\nu}_{\omega n}$. For $x = 0$ (64) becomes

$$0^{\nu}_{\omega n} = \left[\alpha - (\nu-1)\,\omega\right] 0^{\nu-1}_{\omega n} - 2\,\alpha\,\frac{\omega+1}{\nu} \sum_{s=2}^{\nu} \binom{\nu}{s} \frac{0^{s}_{\omega 1}}{\omega^2 - 1}\, 0^{\nu-s}_{\omega n} \qquad (65)$$

which was actually used for checking the results obtained obtained above by (32).

By specifying the polynomial Q in (60) we may evidently obtain any number of formulas for calculating our polynomials by recurrence.

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