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A THEOREM CONCERNING SERIES OF POSITIVE TERMS, WITH APPLICATIONS TO THE THEORY OF FUNCTIONS

BY

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1.

It was first proved by Koebe[1] that if

$$w = w(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{1.1}$$

is regular and schlicht for

$$r = |z| < 1, \tag{1,11}$$

that is to say if the transformation $w = w(z)$ effects a (1, 1) representation of the unit circle in the z-plane on a domain in the w-plane, then

$$|w(z)| \leqq \varphi(r) \qquad (0 < r < 1) \tag{1.2}$$

and

$$|a_n| \leqq \alpha(n), \tag{1.3}$$

where φ and α are functions respectively of r and n alone.

It was afterwards shown by Bieberbach[2] that

$$|w(z)| \leqq \frac{r}{(1-r)^2}, \tag{1.4}$$

[1] The results (1.2) and (1.3) are included in Koebe's 'Verzerrungssatz': see P. Koebe, 'Über die Uniformisierung der algebraischen Kurven durch automorphe Funktionen mit imaginärer Substitutionsgruppe', *Göttinger Nachrichten*, 1909, 68—76 (73). For proofs see Landau's book *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, § 27.

[2] L. Bieberbach, 'Zwei Sätze über das Verhalten analytischer Funktionen in der Umgebung wesentlich singulärer Stellen', *Mathematische Zeitschrift*, 2 (1918), 158—170 (161). The actual results (1.4) and (1.5) appear here for the first time, but (1.4) is, as Bieberbach points out, merely the result of combining those of two earlier memoirs, viz: L. Bieberbach, 'Über die schlichte Abbildung des Einheitskreises', *Berliner Sitzungsberichte*, 1916, 940—955, and G. Pick, 'Über den Koebeschen Verzerrungssatz', *Leipziger Berichte*, 68 (1916), 58—64.

this result being, as is shown by the example

$$w(z) = \frac{z}{(1-z)^2}, \tag{1.41}$$

the best possible of its kind; and that

$$|a_n| < An^2, \tag{1.5}$$

where A is an absolute constant. Finally Littlewood [1] proved that

$$|a_n| < An, \tag{1.6}$$

which is a best possible result apart from the constant factor A.

If w is schlicht, then

$$\sum_1^\infty n|a_n|^2 r^{2n} \leqq \left(\sum_1^\infty |a_n| r^n\right)^2 \quad (r < 1), \tag{1.7}$$

since the left hand side is, when multiplied by π, the area of the image of the circle $|z| \leqq r$, and this cannot exceed

$$\pi (\operatorname*{Max}_{|z| \leqq r} |w|)^2 \leqq \pi\left(\sum_1^\infty |a_n| r^n\right)^2.$$

It is natural to ask whether the theorems which we have quoted are corollaries of (1.7) alone.

We cannot expect an entirely affirmative answer, since (1.7) expresses part only of the data. It will be found, however, that a great deal, and in particular the results of Koebe, is directly deducible from (1.7). Our first object is to prove this, and so to reduce to the absolute minimum the amount of genuine function theory demanded by the proof of Koebe's theorems. We are then led naturally to

[1] J. E. Littlewood, 'On inequalities in the theory of functions', *Proc. London Math. Soc.* (2), 23 (1925), 481—419 (499); see also *ibid.* v—ix (*Records* for 8 November, 1923).

developments in other directions, in particular concerning functions which assume a given value at most a given number of times.

2.

2.1. Theorem 1. Suppose that

$$b_0 = 0, \quad b_n \geqq 0 \quad (n > 0),$$

that $\sum b_n r^n$ is convergent for $r < 1$, and that

$$\sum' n b_n^2 r^{2n} \leqq p\left(\sum' b_n r^n\right)^2, \tag{2.11}$$

where $p \geqq 1$, for $r < 1$. Then

$$b_n < A(p)\, n^{\beta(p)} \mu, \tag{2.12}$$

where

$$\mu = \operatorname{Max}(b_1, b_2, \ldots, b_{[p]}), \tag{2.121}$$

and A and β are functions of p only.

If we write (2.11) in the form

$$\sum' n b_n^2 r^{2n} \leqq p \sum' \sum' b_m b_n r^{m+n},$$

divide by r, and integrate over the range $(0, r)$, we obtain

$$\sum' b_n^2 r^{2n} \leqq 2p \sum' \sum' \frac{b_m b_n}{m+n} r^{m+n}.$$

Repeating this process $2k+1$ times, and writing

$$c_n = n^{-k} b_n r^n, \quad u(m, n) = \frac{1}{m+n}\left(\frac{4mn}{(m+n)^2}\right)^k, \tag{2.13}$$

we find

$$\begin{aligned} \sum' c_n^2 &\leqq 2p \sum' \sum' c_m c_n u(m, n) \\ &= p \sum' \frac{c_n^2}{n} + 4p \sum_{n > m > 0}\sum' c_m c_n u(m, n). \end{aligned} \tag{2.14}$$

We now write ϱ for $[p]$, and M generally for a number of the type $A(p)\mu$. We have then, from (2.14),

$$\left(1-\frac{p}{\varrho+1}\right)\sum_{\varrho+1}^{\infty} c_n^2 \leqq \sum c_n^2 - p\sum_{\varrho+1}^{\infty}\frac{c_n^2}{n}$$

$$\leqq p\sum_{1}^{\varrho}\frac{c_n^2}{n} + A(p)S < M^2 + A(p)S$$

(2.15) $$\sum c_n^2 < M^2 + \sum_{\varrho+1}^{\infty} c_n^2 < M^2 + \frac{\varrho+1}{\varrho+1-p}\left(M^2 + A(p)S\right)$$

$$< M^2 + A(p)S,$$

where

(1.151) $$S = \sum_{n>m>0}\sum c_m c_n u(m, n).$$

But

(2.16) $$2S \leqq \sum_{n>m>0}\sum (c_m^2 + c_n^2)\, u(m, n) = \sum c_m^2 V_m + \sum c_n^2 W_n,$$

where

(2.161) $$V_m = \sum_{m+1}^{\infty} u(m, n) < \int_m^{\infty} u(m, x)\, dx = \int_1^{\infty}\left(\frac{4u}{(1+u)^2}\right)^k \frac{du}{1+u},$$

(2.162) $$W_n = \sum_{1}^{n-1} u(m, n) < \int_0^{m} u(y, n)\, dy = \int_0^{1}\left(\frac{4u}{(1+u)^2}\right)^k \frac{du}{1+u},$$

and each of these integrals tends to zero when $k \to \infty$. We have therefore

(2.17) $$A(p)S < A(p)\eta(k)\sum c_n^2$$

where $\eta(k)$ is a function of k only which tends to zero when $k \to \infty$. We can choose $k = k(p)$ so that

(2.18) $$A(p)\eta(k) < \frac{1}{2};$$

and we have then from (2. 15), (2. 17) and (2. 18),

$$\sum{}' c_n^2 < M^2 + \frac{1}{2} \sum{}' c_n^2 ,$$

$$c_n^2 < \sum{}' c_n^2 < M^2 ,$$

$$b_n r^n < M n^k .$$

This is true for all values of r less than 1. We may therefore make $r \to 1$, when we obtain (2. 12), with $k(p)$ for $\beta(p)$.

2. 2. If we take $b_1 = 1$, $b_n = |a_n|$, and $p = 1$, so that (2. 17) reduces to (1. 7), we obtain

$$|a_n| < A n^{\beta} , \quad |w(z)| < \frac{A}{(1-r)^{\beta}} ,$$

where the A's and β's (different in the two inequalities) are now absolute constants. These inequalities include Koebe's (1. 2) and (1. 3). More generally, if we suppose that $w(z)$ assumes no value more than p times, (2. 11) holds with $b_n = |a_n|$. We thus obtain

Theorem 2. If $w(z) = \sum_{1}^{\infty} a_n z^n$ is regular for $r < 1$, and assumes no value more than p times, then

$$|a_n| < A(p) n^{\beta(p)} \mu ,$$

where

$$\mu = \text{Max}\,(|a_1|, |a_2|, \ldots, |a_p|).$$

3.

3. 1. It is convenient at this point to make a digression into the theory of definite integrals.

Suppose that $a > 0$, $f(x) \geqq 0$, and that the integral

$$\int_a^\infty f(x)\, e^{-\delta x}\, dx$$

is convergent for all positive values of δ. Then the inequality corresponding to (2. 11) is

$$(3.\,11)\qquad \int_a^\infty x f^2(x)\, e^{-2\delta x} dx \leqq p\left(\int_a^\infty f(x)\, e^{-\delta x}\, dx\right)^2,$$

where p is now any positive number. We suppose first that (3. 11) is true for every positive δ. We write $\varphi = f$ if $x \geqq a$, $\varphi = 0$ if $x < a$; and (3. 11) may then be written in the form

$$(3.\,12)\qquad \int_0^\infty x\,\varphi^2(x)\, e^{-2\delta x} dx \leqq p \int_0^\infty\!\!\int_0^\infty \varphi(x)\,\varphi(y)\, e^{-\delta x - \delta y} dx\, dy.$$

We write ζ for δ in (3. 12), multiply by

$$\frac{(\zeta - \delta)^{2k}}{(2k+1)!},$$

where k is now any number greater than $-\frac{1}{2}$, and integrate with respect to ζ over the range (δ, ∞). Observing that

$$\frac{1}{\Gamma(2k+1)} \int_\delta^\infty (\zeta - \delta)^{2k}\, e^{-2\zeta x}\, d\zeta = (2x)^{-2k-1}\, e^{-2\delta x},$$

and writing

$$(3.\,13)\qquad \psi(x) = x^{-k}\varphi(x)\, e^{-\delta x}, \qquad u(x, y) = \frac{1}{x+y}\left(\frac{4xy}{(x+y)^2}\right)^k,$$

we obtain

$$(3.14) \quad \int_0^\infty \psi^2(x)\,dx \leq 2p \int_0^\infty\int_0^\infty \psi(x)\,\psi(y)\,u(x,y)\,dx\,dy,$$

the inequality corresponding to (2. 14).

We have now

$$\int_0^\infty\int_0^\infty \psi(x)\,\psi(y)\,u(x,y)\,dx\,dy$$

$$= \int_0^\infty\int_0^\infty \psi(x)\left(\frac{x}{y}\right)^{\frac{1}{4}}\sqrt{u(x,y)}\cdot\psi(y)\left(\frac{y}{x}\right)^{\frac{1}{4}}\sqrt{u(y,x)}\,dx\,dy$$

$$\leq \frac{1}{2}\int_0^\infty\int_0^\infty \psi^2(x)\sqrt{\frac{x}{y}}\,u(x,y)\,dx\,dy + \frac{1}{2}\int_0^\infty\int_0^\infty \psi^2(y)\sqrt{\frac{y}{x}}\,u(y,x)\,dy\,dx$$

$$= \int_0^\infty\int_0^\infty \psi^2(x)\sqrt{\frac{x}{y}}\,u(x,y)\,dx\,dy = \int_0^\infty \psi^2(x)\,P(x)\,dx,$$

where

$$P(x) = \int_0^\infty \sqrt{\frac{x}{y}}\,u(x,y)\,dy = \int_0^\infty \left(\frac{4u}{(1+u)^2}\right)^k \frac{du}{(1+u)\sqrt{u}} = J_k$$

is independent of x. Thus (3. 14) gives

$$(3.15) \qquad 1 \leq 2p J_k.$$

But, putting $u = e^{2z}$, we obtain

$$(3.16) \quad J_k = \int_{-\infty}^\infty (ch\,z)^{-2k-1}\,dz = \sqrt{\pi}\,\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} = O\left(\frac{1}{\sqrt{k}}\right)$$

for large values of k. Thus (3. 15) involves a contradiction if k is sufficiently large, so that it is impossible that (3. 11) should be true for all positive values of δ; and the analogy between series and integrals so far fails.

3. 2. There is in fact a theorem for integrals analogous to Theorem 1, but before proving this we may point out a curious corollary of the analysis of § 3. 1. If $f(x)$ is any

function (real or complex) such that $f(x) = O(e^{\varepsilon x})$ for every positive ε, then the function

$$F(z) = \int_a^\infty f(x)\, e^{-zx}\, dx,$$

where $a > 0$, is regular for $R(z) > 0$, and the area of the image, for $F(z)$, of the half-plane $R(z) > \delta$ is

$$\pi \int_a^\infty x\, |f(x)|^2\, e^{-2\delta x}\, dx$$

areas covered more than once being counted multiply. Suppose that $F(z)$ assumes no value more than p times in the half plane $R(z) > 0$. Then it follows, as with series, that

$$\int_a^\infty x\, |f(x)|^2\, e^{-2\delta x}\, dx \leqq \left(\int_a^\infty |f(x)|\, e^{-\delta x}\, dx\right)^2$$

for $\delta > 0$; and this we have seen to be impossible. Thus we obtain

Theorem 3. If $f(x) = O(e^{\varepsilon x})$ for every positive ε, then the function

$$F(z) = \int_a^\infty f(x)\, e^{-zx}\, dx \qquad (a > 0)$$

cannot be schlicht, or assume no value more than p times, in the half plane $R(z) > 0$.

It is essential here that a be positive. The consideration of a simple example will help to elucidate the result. Suppose that

$$a = 1, \quad f(x) = 1\ (x \geqq 1), \quad F(z) = \frac{e^{-z}}{z}.$$

Then the equation

$$e^{-z} = \gamma z$$

must have, for appropriate values of γ, more than any assigned number p of zeros in the positive half plane; and it is easy to verify that this is so. Suppose for example that γ is positive and small. There is then one root which is large and positive, and an infinity of complex roots. These roots lie on a curve, symmetrical about the real axis, which crosses the imaginary axis at a considerable distance, and proceeds to infinity in the negative half plane, in a direction which approximates to parallelism with the imaginary axis. The number of roots in the positive half plane is finite for every γ, but tends to infinity when $\gamma \to 0$.[1]

The function

$$\int_0^\infty e^{-zx}\,dx = \frac{1}{z},$$

on the other hand, is schlicht in $R(z) > 0$.

3.2. In § 3.1 we supposed that (3.11) was true for every positive δ. We shall now assume less, viz., that

$$(3.21) \qquad \int_a^\infty x f^2(x)\, e^{-2\delta x}\,dx < (p+\varepsilon)\left(\int_a^\infty f(x)\, e^{-\delta x}\,dx\right)^2$$

for every positive ε and sufficiently small values of δ, that is to say for

$$(2.211) \qquad 0 < \delta \leqq \delta_0(\varepsilon);$$

or, in other words, that

$$(3.22) \qquad \overline{\lim_{\delta \to 0}} \left(\int_a^\infty x f^2(x)\, e^{-\delta x}\,dx \Bigg/ \left(\int_a^\infty f(x)\, e^{-\delta x}\,dx\right)^2\right) \leqq p.$$

[1] For accurate information about the zeros of this and more general transcendental equations of similar type, see G. H. Hardy, 'The asymptotic solution of certain transcendental equations', *Quarterly Journal of Mathematics*, 35 (1904), 261-282; and E. Schwengeler, 'Geometrisches über die Verteilung der Nullstellen spezieller ganzer Funktionen', *Dissertation*, Zürich, 1925.

In this case we can deduce a conclusion analogous to that of Theorem 1.

If the integral

$$\int_a^\infty f(x)\,dx \tag{3.23}$$

is convergent then

$$\int_a^\infty x f^2(x)\,dx \tag{3.24}$$

is convergent, by (3.21). Suppose next that (3.23) is divergent. We may integrate (3.21) or (3.22) over the range (δ, ∞), any number $2k+1$ of times, and assert the resultant inequality in the same sense as the original inequality, *i. e.* for sufficiently small values of δ, provided that the integral which then appears on the right hand side of the inequality tends to infinity when $\delta \to 0$ [1], or, *a fortiori*, provided that the integral

$$\int_0^\infty x^{-2k} f^2(x)\,dx \tag{3.25}$$

is divergent. More generally we may, under the same condition, operate on (3.21) as we operated on (3.11) in § 3. 1. We thus obtain the analogue of (3.14), viz.,

[1] From

$$\overline{\lim_{\delta \to 0}} \frac{g(\delta)}{h(\delta)} \leqq p,$$

where g and h are positive, we can deduce

$$\overline{\lim_{\delta \to 0}} \left(\int_\delta^\infty g(\zeta)\,d\zeta \Big/ \int_\delta^\infty h(\zeta)\,d\zeta\right) \leqq p,$$

provided

$$\int_\delta^\infty h(\zeta)\,d\zeta \to \infty$$

when $\delta \to 0$.

$$\int_0^\infty \psi^2(x)\,dx < 2(p+\varepsilon)\int_0^\infty\int_0^\infty \psi(x)\,\psi(y)\,u(x,y)\,dx\,dy$$

for $0 < \delta \leqq \delta_k(\varepsilon)$. From this we obtain

$$1 < 2(p+\varepsilon)J_k,$$

as in § 3. 1. The equation

$$2p\sqrt{\pi}\,\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)} = 1 \tag{3. 26}$$

has just one real root $k = k_p$ greater than $-\frac{1}{2}$, and large when p is large, and, if $k > k_p$, we obtain a contradiction. We thus obtain

Theorem 4. *If $a > 0$, $f(x) \geqq 0$, the integral (3. 23) is divergent, and the inequality (3. 22) is satisfied, then the integral*

$$\int_a^\infty \left(x^{-k} f(x)\right)^2 dx \tag{3. 27}$$

is convergent for all values of k greater than k_p, the root, greater than $-\frac{1}{2}$, of the equation (3. 25).

The theorem is true, but trivial, when the integral (3. 23) is convergent, since (3. 22) then involves the convergence of (3. 24), and $k_p > -\frac{1}{2}$.

3. 3. It is interesting to observe that the result of Theorem 4 is, in certain senses, the best possible obtainable.

In the first place, it is plainly not possible to deduce from the hypotheses a conclusion of the type

$$f(x) = O(x^l);$$

for the truth of (3. 22) vould not be affected by adding to f any function g such that the integrals

$$\int_a^\infty x g^2(x)\,dx\,, \quad \int_a^\infty g(x)\,dx$$

are convergent, and this function might have, for appropriate sequences of values of x, any order of magnitude we please.

A more important point is that, in Theorem 4, the number k_p cannot be replaced by any smaller number.

To prove this, suppose that $f = x^{k-\frac{1}{2}}$. Then

$$\int_a^\infty f(x)\,e^{-\delta x}\,dx \sim \int_0^\infty x^{k-\frac{1}{2}}\,e^{-\delta x}\,dx = \Gamma\left(k+\frac{1}{2}\right)\delta^{-k-\frac{1}{2}},$$

$$\int_a^\infty x f^2(x)\,e^{-2\delta x}\,dx \sim \int_0^\infty x^{2k}\,e^{-2\delta x}\,dx = \Gamma(2k+1)\,(2\delta)^{-2k-1},$$

when $\delta \to 0$, and (3. 22) is satisfied if

$$2^{-2k-1}\,\Gamma(2k+1) \leqq p\,\Gamma^2\left(k+\frac{1}{2}\right),$$

or

$$2p\sqrt{\pi}\,\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)} \geqq 1\,, \quad k \leqq k_p\,.$$

But the integral (3. 26) is divergent.

4.

4.1. There is a theorem for series which corresponds exactly to Theorem 4, viz.,

Theorem 5. If $b_n \geqq 0$, $\sum b_n$ is divergent, $\sum b_n r^n$ is convergent for $r<1$, and

$$\overline{\lim_{r\to 1}}\left(\sum n b_n^2 r^{2n} \Big/ \left(\sum b_n r^n\right)^2\right) \leqq p, \tag{4.11}$$

where $p>0$, then

$$\sum (n^{-k} b_n)^2 \tag{4.12}$$

is convergent for $k>k_p$. In this proposition, the number k_p cannot be replaced by any smaller number,

We need only sketch the proof, which follows generally the lines of that of Theorem 4. We prove, substantially as in § 3.2, that, if (4.12) is divergent, and c_n and $u(m, n)$ are defined as in (2.13), then

$$\sum c_n^2 < 2(p+\varepsilon)\sum\sum c_m c_n u(m, n)$$

for $r_k(\varepsilon) \leqq r < 1$. We have now

$$\sum\sum c_m c_n u(m, n)$$

$$=\sum\sum c_m \left(\frac{m}{n}\right)^{\frac{1}{4}}\sqrt{u(m, n)}\cdot c_n\left(\frac{n}{m}\right)^{\frac{1}{4}}\sqrt{u(n, m)}$$

$$\leqq \frac{1}{2}\sum\sum c_m^2\sqrt{\frac{m}{n}}\, u(m, n)+\frac{1}{2}\sum\sum c_m^2\sqrt{\frac{n}{m}}\, u(n, m)$$

$$=\sum\sum c_m^2\sqrt{\frac{m}{n}}\, u(m, n)=\sum c_m^2 P_m,$$

where

$$P_m=\sum_{n=1}^{\infty}\sqrt{\frac{m}{n}}\, u(m, n)=\sum_{n=1}^{\infty} u\left(1, \frac{n}{m}\right)\sqrt{\frac{m}{n}}\frac{1}{m};$$

and so

$$(4.13) \qquad \sum_{n=1}^{\infty}{}' \left(1 - 2(p+\varepsilon)P_m\right) c_m^2 \leqq 0.$$

Now when $m \to \infty$,

$$P_m \to \int_0^\infty u(1, y) \frac{dy}{\sqrt{y}} = J_k = \sqrt{\pi} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)} < \frac{1}{2p}$$

if $k > k_p$; and then

$$(4.14) \qquad 1 - 2(p+\varepsilon) P_m > \eta > 0$$

for sufficiently large values of m. This contradicts (4.13) if (4.12) is divergent.

That k_p is the smallest number which can occur in Theorem 5 may be shown by what is substantially the example of § 3.3.

5.

5.1. Our original proof of our principal theorem (Theorem 1) has undergone a series of simplifications since the thorem was discovered, in which we have been assisted by the observations of friends to whom we had communicated it. In particular Prof. E. Landau sent us, in a letter dated 3 Feb. 1925, a proof of the theorem, for $p = 1$, which is considerably simpler than any which we then possessed, and is in principle the same as that of § 2. He also remarked that one of the essential improvements in the proof had been suggested to him by Prof. I. Schur. Finally we should add that Theorem 5 was pointed out to us by Prof. G. Pólya, to whom we had communicated Theorem 4, and that we have improved our proof of the latter theorem by assimilating it to Pólya's proof of the former, which is that which we give in § 4.

Færdig fra Trykkeriet den 31. December 1925.